

Bohr's absolute convergence problem for \mathcal{H}_p -Dirichlet series in Banach spaces

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Abstract

The Bohr-Bohnenblust-Hille Theorem states that the width of the strip in the complex plane on which an ordinary Dirichlet series $\sum_n a_n n^{-s}$ converges uniformly but not absolutely is less than or equal to $1/2$, and this estimate is optimal. Equivalently, the supremum of the absolute convergence abscissas of all Dirichlet series in the Hardy space \mathcal{H}_∞ equals $1/2$. By a surprising fact of Bayart the same result holds true if \mathcal{H}_∞ is replaced by any Hardy space \mathcal{H}_p , $1 \leq p < \infty$, of Dirichlet series. For Dirichlet series with coefficients in a Banach space X the maximal width of Bohr's strips depend on the geometry of X ; Defant, García, Maestre and Pérez-García proved that such maximal width equal $1 - 1/\text{Cot}(X)$, where $\text{Cot}(X)$ denotes the maximal cotype of X . Equivalently, the supremum over the absolute convergence abscissas of all Dirichlet series in the vector-valued Hardy space $\mathcal{H}_\infty(X)$ equals $1 - 1/\text{Cot}(X)$. In this article we show that this result remains true if $\mathcal{H}_\infty(X)$ is replaced by the larger class $\mathcal{H}_p(X)$, $1 \leq p < \infty$.

1 Main result and its motivation

Given a Banach space X , an ordinary Dirichlet series in X is a series of the form $D = \sum_n a_n n^{-s}$, where the coefficients a_n are vectors in X and s is a complex variable. Maximal domains where such Dirichlet series converge conditionally, uniformly or absolutely are half planes $[\text{Re} > \sigma]$, where $\sigma = \sigma_c, \sigma_u$ or σ_a are called the abscissa of conditional, uniform or absolute convergence, respectively. More precisely, $\sigma_\alpha(D)$ is the infimum of all $r \in \mathbb{R}$ such that on $[\text{Re} > r]$ we have convergence of D of the requested type $\alpha = c, u$ or a . Clearly, we have $\sigma_c(D) \leq \sigma_u(D) \leq \sigma_a(D)$, and it can be easily shown that $\sup \sigma_a(D) - \sigma_c(D) = 1$, where the supremum is taken over all Dirichlet series D with coefficients in X . To determine the maximal width of the strip on which a Dirichlet series in X converges uniformly

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but not absolutely, is more complicated. The main result of [8] states, with the notation given below, that

$$S(X) := \sup \sigma_a(D) - \sigma_u(D) = 1 - \frac{1}{\text{Cot}(X)}. \quad (1)$$

Recall that a Banach space X is of cotype q , $2 \leq q < \infty$ whenever there is a constant $C \geq 0$ such that for each choice of finitely many vectors $x_1, \dots, x_N \in X$ we have

$$\left(\sum_{k=1}^N \|x_k\|_X^q \right)^{1/q} \leq C \left(\int_{\mathbb{T}^N} \left\| \sum_{k=1}^N x_k z_k \right\|_X^2 dz \right)^{1/2}, \quad (2)$$

where $\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$ and \mathbb{T}^N is endowed with N th product of the normalized Lebesgue measure on \mathbb{T} . We denote by $C_r(X)$ the best of such constants C . As usual we write

$$\text{Cot}(X) := \inf \left\{ 2 \leq q < \infty \mid X \text{ cotype } q \right\},$$

and (although this infimum in general is not attained) we call it the optimal cotype of X . If there is no $2 \leq q < \infty$ for which X has cotype q , then X is said to have no finite cotype, and we put $\text{Cot}(X) = \infty$. To see an example,

$$\text{Cot}(X)(\ell_q) = \begin{cases} q & \text{for } 2 \leq q \leq \infty \\ 2 & \text{for } 1 \leq q \leq 2. \end{cases}$$

The scalar case $X = \mathbb{C}$ in (1) was first studied by Bohr and Bohnenblust-Hille: In 1913 Bohr in [4] proved that $S(\mathbb{C}) \leq \frac{1}{2}$, and in 1931 Bohnenblust and Hille in [3] that $S(\mathbb{C}) \geq \frac{1}{2}$. Clearly, the equality

$$S(\mathbb{C}) = \frac{1}{2}, \quad (3)$$

nowadays called *Bohr-Bohnenblust-Hille Theorem*, fits with (1). Let us give a second formulation of (1). Define the vector space $\mathcal{H}_\infty(X)$ of all Dirichlet series $D = \sum_n a_n n^{-s}$ in X such that

- $\sigma_c(D) \leq 0$,
- the function $D(s) = \sum_n a_n \frac{1}{n^s}$ on $\text{Re } s > 0$ is bounded.

Then $\mathcal{H}_\infty(X)$ together with the norm

$$\|D\|_{\mathcal{H}_\infty(X)} = \sup_{\text{Re } s > 0} \left\| \sum_{n=1}^{\infty} a_n \frac{1}{n^s} \right\|_X$$

forms a Banach space. For any Dirichlet series D in X we have

$$\sigma_u(D) = \inf \left\{ \sigma \in \mathbb{R} \mid \sum_n \frac{a_n}{n^\sigma} \frac{1}{n^s} \in \mathcal{H}_\infty(X) \right\}. \quad (4)$$

In the scalar case $X = \mathbb{C}$, this is (what we call) *Bohr's fundamental theorem* from [5], and for Dirichlet series in arbitrary Banach spaces the proof follows similarly. Together with (4) a simply translation argument gives the following reformulation of (1):

$$S(X) = \sup_{D \in \mathcal{H}_\infty(X)} \sigma_a(D) = 1 - \frac{1}{\text{Cot}(X)}. \quad (5)$$

Following an ingenious idea of Bohr each Dirichlet series may be identified with a power series in infinitely many variables. More precisely, fix a Banach space X and denote by $\mathfrak{P}(X)$ the vector space of all formal power series $\sum_{\alpha} c_{\alpha} z^{\alpha}$ in X and by $\mathfrak{D}(X)$ the vector space of all Dirichlet series $\sum_n a_n n^{-s}$ in X . Let as usual $(p_n)_n$ be the sequence of prime numbers. Since each integer n has a unique prime number decomposition $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k} = p^{\alpha}$ with $\alpha_j \in \mathbb{N}_0$, $1 \leq j \leq k$, the linear mapping

$$\mathfrak{B}_X : \mathfrak{P}(X) \longrightarrow \mathfrak{D}(X), \quad \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_{\alpha} z^{\alpha} \rightsquigarrow \sum_{n=1}^{\infty} a_n n^{-s}, \quad \text{where } a_{p^{\alpha}} = c_{\alpha} \quad (6)$$

is bijective; we call \mathfrak{B}_X the *Bohr transform in X* . As discovered by Bayart in [1] this (*a priori* very) formal identification allows to develop a theory of Hardy spaces of scalar-valued Dirichlet series.

Similarly we now define Hardy spaces of X -valued Dirichlet series. Denote by dw the normalized Lebesgue measure on the infinite dimensional polytorus $\mathbb{T}^{\infty} = \prod_{k=1}^{\infty} \mathbb{T}$, e.g. the countable product measure of the normalized Lebesgue measure on \mathbb{T} . For any multi index $\alpha = (\alpha_1, \dots, \alpha_n, 0, \dots) \in \mathbb{Z}^{(\mathbb{N})}$ (all finite sequences in \mathbb{Z}) the α th Fourier coefficient $\hat{f}(\alpha)$ of $f \in L_1(\mathbb{T}^{\infty}, X)$ is given by

$$\hat{f}(\alpha) = \int_{\mathbb{T}^{\infty}} f(w) w^{-\alpha} dw,$$

where we as usual write w^{α} for the monomial $w_1^{\alpha_1} \cdots w_n^{\alpha_n}$. Then, given $1 \leq p < \infty$, the X -valued Hardy space on \mathbb{T}^{∞} is the subspace of $L_p(\mathbb{T}^{\infty}, X)$ defined as

$$H_p(\mathbb{T}^{\infty}, X) = \left\{ f \in L_p(\mathbb{T}^{\infty}, X) \mid \hat{f}(\alpha) = 0, \quad \forall \alpha \in \mathbb{Z}^{(\mathbb{N})} \setminus \mathbb{N}_0^{(\mathbb{N})} \right\}. \quad (7)$$

Assigning to each $f \in H_p(\mathbb{T}^{\infty}, X)$ its unique formal power series $\sum_{\alpha} \hat{f}(\alpha) z^{\alpha}$ we may consider $H_p(\mathbb{T}^{\infty}, X)$ as a subspace of $\mathfrak{P}(X)$. We denote the image of this subspace under the Bohr transform \mathfrak{B}_X by

$$\mathcal{H}_p(X).$$

This vector space of all (so-called) $\mathcal{H}_p(X)$ -Dirichlet series D together with the norm

$$\|D\|_{\mathcal{H}_p(X)} = \|\mathfrak{B}_X^{-1}(D)\|_{H_p(\mathbb{T}^{\infty}, X)}$$

forms a Banach space; in other words, through Bohr's transform \mathfrak{B}_X from (6) we by definition identify

$$\mathcal{H}_p(X) = H_p(\mathbb{T}^{\infty}, X), \quad 1 \leq p < \infty.$$

For $p = \infty$ we this way of course could also define a Banach space $\mathcal{H}_{\infty}(X)$, and it turns out that at least in the scalar case $X = \mathbb{C}$ this definition then coincides with the one given above; but we remark that these two $\mathcal{H}_{\infty}(X)$'s are different for arbitrary X . It is important to note that by the Birkhoff-Khinchine ergodic theorem the following internal description of the $\mathcal{H}_p(X)$ -norm for finite Dirichlet polynomials $D = \sum_{k=1}^n a_k n^{-s}$ holds:

$$\|D\|_{\mathcal{H}_p(X)} = \lim_{T \rightarrow \infty} \left(\frac{1}{2T} \int_{-T}^T \left\| \sum_{k=1}^n a_k \frac{1}{n^t} \right\|_X^p dt \right)^{1/p}$$

(see e.g. Bayart [1] for the scalar case, and the vector-valued case follows exactly the same way).

Motivated by (4) we define for $D \in \mathfrak{D}(X)$ and $1 \leq p < \infty$

$$\sigma_{\mathcal{H}_p(X)}(D) := \inf \left\{ \sigma \in \mathbb{R} \mid \sum_n \frac{a_n}{n^\sigma} \frac{1}{n^s} \in \mathcal{H}_p(X) \right\},$$

and motivated by (5) we define

$$S_p(X) := \sup_{D \in \mathfrak{D}(X)} \sigma_a(D) - \sigma_{\mathcal{H}_p(X)}(D) = \sup_{D \in \mathcal{H}_p(X)} \sigma_a(D)$$

(for the second equality use again a simple translation argument). A result of Bayart [1] shows that for every $1 \leq p < \infty$

$$S_p(\mathbb{C}) = \frac{1}{2}, \quad (8)$$

which according to Helson [13] is a bit surprising since $\mathcal{H}_\infty(\mathbb{C})$ is much smaller than $\mathcal{H}_p(\mathbb{C})$.

The following theorem unifies and generalizes (1), (3) as well as (8), and it is our main result.

Theorem 1.1. *For every $1 \leq p \leq \infty$ and every Banach space X we have*

$$S_p(X) = 1 - \frac{1}{\text{Cot}(X)}.$$

The proof will be given in section 4. But before we start let us give an interesting reformulation in terms of the monomial convergence of X -valued H_p -functions on \mathbb{T}^∞ . Fix a Banach space X and $1 \leq p \leq \infty$, and define the set of monomial convergence of $H_p(\mathbb{T}^\infty, X)$:

$$\text{mon } H_p(\mathbb{T}^\infty, X) = \left\{ z \in B_{c_0} \mid \sum_\alpha \|\hat{f}(\alpha) z^\alpha\|_X < \infty \text{ for all } f \in H_p(\mathbb{T}^\infty, X) \right\}.$$

Philosophically, this is the largest set M on which for each $f \in H_p(\mathbb{T}^\infty, X)$ the definition $g(z) = \sum_\alpha \hat{f}(\alpha) z^\alpha$, $z \in M$ leads to an extension of f from the distinguished boundary \mathbb{T}^∞ to its “interior” B_{c_0} (the open unit ball of the Banach space c_0 of all null sequences). For a detailed study of sets of monomial convergence in the scalar case $X = \mathbb{C}$ see [9], and in the vector-valued case [10].

We later need the following two basic properties of monomial domains (in the scalar case see [8, p.550] and [7, Lemma 4.3], and in the vector-valued case the proofs follow similar lines).

Remark 1.2.

- (1) Let $z \in \text{mon } H_p(\mathbb{T}^\infty, X)$. Then $u = (z_{\sigma(n)})_n \in \text{mon } H_p(\mathbb{T}^\infty, X)$ for every permutation σ of \mathbb{N} .
- (2) Let $z \in \text{mon } H_p(\mathbb{T}^\infty, X)$ and $x = (x_n)_n \in \mathbb{D}^\infty$ be such that $|x_n| \leq |z_n|$ for all but finitely many n 's. Then $x \in \text{mon } H_p(\mathbb{T}^\infty, X)$.

Given $1 \leq p \leq \infty$ and a Banach space X , the following number measures the size of $\text{mon } H_p(\mathbb{T}^\infty, X)$ within the scale of ℓ_r -spaces:

$$M_p(X) = \sup \left\{ 1 \leq r \leq \infty \mid \ell_r \cap B_{c_0} \subset \text{mon } H_p(\mathbb{T}^\infty, X) \right\}.$$

The following result is a reformulation of Theorem 1.1 in terms of vector-valued H_p -functions on \mathbb{T}^∞ through Bohr's transform \mathfrak{B}_X . The proof is modelled along ideas from Bohr's seminal article [4, Satz IX].

Corollary 1.3. *For each Banach space X and $1 \leq p \leq \infty$ we have*

$$M_p(X) = \frac{\text{Cot}(X)}{\text{Cot}(X) - 1}.$$

Proof. We are going to prove that $S_p(X) = 1/M_p(X)$, and as a consequence the conclusion follows from Theorem 1.1. We begin by showing that $S_p(X) \leq 1/M_p(X)$. We fix $q < M_p(X)$ and $r > 1/q$; then we have that $(\frac{1}{p_n^r})_n \in \ell_q \cap B_{c_0}$ and, by the very definition of $M_p(X)$, $\sum_\alpha \|\hat{f}(\alpha) (\frac{1}{p^r})^\alpha\|_X < \infty$ converges absolutely for every $f \in H_p(\mathbb{T}^\infty, X)$. We choose now an arbitrary Dirichlet series

$$D = \mathfrak{B}_X f = \sum_n a_n \frac{1}{n^r} \in \mathcal{H}_p(X) \text{ with } f \in H_p(\mathbb{T}^\infty, X).$$

Then

$$\sum_n \|a_n\|_X \frac{1}{n^r} = \sum_\alpha \|a_{p^\alpha}\|_X \left(\frac{1}{p^\alpha}\right)^r = \sum_\alpha \|\hat{f}(\alpha)\|_X \left(\frac{1}{p^r}\right)^\alpha < \infty.$$

Clearly, this implies that $S_p(X) \leq r$. Since this holds for each $r > 1/q$, we get that $S_p(X) \leq 1/q$, and since this now holds for each $q < M_p(X)$, we have $S_p(X) \leq 1/M_p(X)$. Conversely, let us take some $q > M_p(X)$; then there is $z \in \ell_q \cap B_{c_0}$ and $f \in H_\infty(\mathbb{T}^\infty, X)$ such that $\sum_\alpha \hat{f}(\alpha) z^\alpha$ does not converge absolutely. By Remark 1.2 we may assume that z is decreasing, and hence $(z_n n^{1/q})_n$ is bounded. We choose now $r > q$ and define $w_n = \frac{1}{p_n^{1/r}}$. By the Prime Number Theorem we know that there is a universal constant $C > 0$ such that

$$0 < \frac{z_n}{w_n} = z_n p_n^{\frac{1}{r}} = z_n n^{\frac{1}{q}} \frac{p_n^{1/r}}{n^{1/q}} = z_n n^{\frac{1}{q}} \left(\frac{p_n}{n}\right)^{\frac{1}{r}} \frac{1}{n^{1/q-1/r}} \leq C z_n n^{\frac{1}{q}} \frac{(\log n)^{1/r}}{n^{1/q-1/r}}.$$

The last term tends to 0 as $n \rightarrow \infty$; hence $z_n \leq w_n$ but for a finite number of n 's. By Remark 1.2 this implies that $\sum_\alpha \hat{f}(\alpha) w^\alpha$ does not converge absolutely. But then $D = \mathfrak{B}_X f = \sum_n a_n n^{-r} \in \mathcal{H}_p(X)$ satisfies

$$\sum_n \|a_n\|_X \frac{1}{n^{1/r}} = \sum_\alpha \|a_{p^\alpha}\|_X \left(\frac{1}{p^{1/r}}\right)^\alpha = \sum_\alpha \|\hat{f}(\alpha)\|_X w^\alpha = \infty.$$

This gives that $\sigma_a(D) \geq 1/r$ for every $r > q$, hence $\sigma_a(D) \geq 1/q$. Since this holds for every $q > M_p(X)$, we finally have $S_p(X) \geq 1/M_p(X)$. \square

We shall use standard notation and notions from Banach space theory, as presented, e.g. in [?, ?]. For everything needed on polynomials in Banach spaces see e.g. [11] and [12].

2 Relevant inequalities

The main aim here is to prove a sort of polynomial extension of the notion of cotype. Recall the definition of $C_q(X)$ from (2). Moreover, from Kahane's inequality we know that, given $1 \leq q < \infty$, there is a (best) constant $K \geq 1$ such that for each Banach space X and each choice finitely many vectors $x_1, \dots, x_N \in X$

$$\left(\int_{\mathbb{T}^N} \left\| \sum_{k=1}^N x_k z_k \right\|_X^2 dz \right)^{1/2} \leq K \int_{\mathbb{T}^N} \left\| \sum_{k=1}^N x_k z_k \right\|_X dz.$$

As usual we write $|\alpha| = \alpha_1 + \dots + \alpha_N$ and $\alpha! = \alpha_1! \dots \alpha_N!$ for every multi index $\alpha \in \mathbb{N}_0^N$.

Proposition 2.1. *Let X be a Banach space of cotype q , $2 \leq q < \infty$, and*

$$P : \mathbb{C}^N \rightarrow X, \quad P(z) = \sum_{\substack{\alpha \in \mathbb{N}_0^N \\ |\alpha|=m}} c_\alpha z^\alpha$$

be an m -homogeneous polynomial. Let

$$T : \mathbb{C}^N \times \dots \times \mathbb{C}^N \rightarrow X, \quad T(z^{(1)}, \dots, z^{(m)}) = \sum_{i_1, \dots, i_m=1}^N a_{i_1, \dots, i_m} z_{i_1}^{(1)} \dots z_{i_m}^{(m)}$$

be the unique m -linear symmetrization of P . Then

$$\left(\sum_{i_1, \dots, i_m} \|a_{i_1, \dots, i_m}\|_X^q \right)^{1/q} \leq (C_q(X) K)^m \frac{m^m}{m!} \int_{\mathbb{T}^N} \|P(z)\|_X dz.$$

Before we give the proof let us note that [?, Theorem 3.2] is an m -linear result that, combined with polarization gives (with the previous notation)

$$\left(\sum_{i_1, \dots, i_m} \|a_{i_1, \dots, i_m}\|_X^q \right)^{1/q} \leq C_q(X)^m \frac{m^m}{m!} \sup_{z \in \mathbb{D}^N} \|P(z)\|.$$

Our result allows to replace (up to the constant K) the $\|\cdot\|_\infty$ norm with the smaller norm $\|\cdot\|_1$. We prepare the proof of Proposition 2.1 with three lemmas.

Lemma 2.2. *Let X be a Banach space of cotype q , $2 \leq q < \infty$. Then for every m -linear form*

$$T : \mathbb{C}^N \times \dots \times \mathbb{C}^N \rightarrow X, \quad T(z^{(1)}, \dots, z^{(m)}) = \sum_{i_1, \dots, i_m=1}^N a_{i_1, \dots, i_m} z_{i_1}^{(1)} \dots z_{i_m}^{(m)}$$

we have

$$\left(\sum_{i_1, \dots, i_m=1}^N \|a_{i_1, \dots, i_m}\|_X^q \right)^{1/q} \leq (C_q(X) K)^m \int_{\mathbb{T}^\infty} \dots \int_{\mathbb{T}^\infty} \|T(z^{(1)}, \dots, z^{(m)})\|_X dz^{(1)} \dots dz^{(m)}.$$

Proof. We prove this result by induction on the degree m . For $m = 1$ the result is an immediate consequence of the definition of cotype q and Kahane's inequality. Assume that

the result holds for $m - 1$. By the continuous Minkowski inequality we then conclude that for every choice of finitely many vectors $a_{i_1, \dots, i_m} \in X$ with $1 \leq i_j \leq N, 1 \leq j \leq m$ we have

$$\begin{aligned}
\sum_{i_1, \dots, i_m} \|a_{i_1, \dots, i_m}\|_X^q &= \sum_{i_1, \dots, i_{m-1}} \sum_{i_m} \|a_{i_1, \dots, i_m}\|_X^q \\
&\leq C_q(X)^q K^q \left(\sum_{i_1, \dots, i_{m-1}} \left(\int_{\mathbb{T}^\infty} \left\| \sum_{i_m} a_{i_1, \dots, i_m} z_{i_m}^{(m)} \right\|_X dz^{(m)} \right)^q \right)^{q/q} \\
&\leq C_q(X)^q K^q \left(\int_{\mathbb{T}^\infty} \left(\sum_{i_1, \dots, i_{m-1}} \left\| \sum_{i_m} a_{i_1, \dots, i_m} z_{i_m}^{(m)} \right\|_X^q \right)^{1/q} dz^{(m)} \right)^q \\
&\leq C_q(X)^{qm} K^{qm} \left(\int_{\mathbb{T}^\infty} \underbrace{\int_{\mathbb{T}^\infty} \dots \int_{\mathbb{T}^\infty}}_{m-1} \left\| \sum_{i_1, \dots, i_{m-1}} a_{i_1, \dots, i_{m-1}} z_{i_1}^{(1)}, \dots, z_{i_{m-1}}^{(m-1)} \right\|_X dz^{(1)} \dots dz^{(m-1)} dz^{(m)} \right)^q,
\end{aligned}$$

which is the conclusion. \square

The following two lemmas are needed to produce a polynomial analog of the preceding result.

Lemma 2.3. *Let X be a Banach space, and $f : \mathbb{C} \rightarrow X$ a holomorphic function. Then for $R_1, R_2, R \geq 0$ with $R_1 + R_2 \leq R$ we have*

$$\int_{\mathbb{T}} \int_{\mathbb{T}} \|f(R_1 z_1 + R_2 z_2)\|_X dz_1 dz_2 \leq \int_{\mathbb{T}} \|f(Rz)\|_X dz.$$

Proof. By the rotation invariance of the normalized Lebesgue measure on \mathbb{T} we get

$$\begin{aligned}
\int_{\mathbb{T}} \int_{\mathbb{T}} \|f(R_1 z_1 + R_2 z_2)\|_X dz_1 dz_2 &= \int_{\mathbb{T}} \int_{\mathbb{T}} \|f(R_1 z_1 z_2 + R_2 z_2)\|_X dz_1 dz_2 \\
&= \int_{\mathbb{T}} \int_{\mathbb{T}} \|f(z_2(R_1 z_1 + R_2))\|_X dz_1 dz_2 = \int_{\mathbb{T}} \int_{\mathbb{T}} \|f(z_2 |R_1 z_1 + R_2|)\|_X dz_2 dz_1 \\
&= \int_{\mathbb{T}} \int_{\mathbb{T}} \|f(z_2 r(z_1) R)\|_X dz_2 dz_1 = \int_0^{2\pi} \int_0^{2\pi} \|f(r(e^{is}) R e^{it})\|_X \frac{dt}{2\pi} \frac{ds}{2\pi}.
\end{aligned}$$

where $r(z) = \frac{1}{R} |R_1 z + R_2|$, $z \in \mathbb{T}$. We know that for each holomorphic function $h : \mathbb{C} \rightarrow X$ we have

$$\int_{\mathbb{T}} \|h(z)\|_X dz = \sup_{0 \leq r \leq 1} \int_0^{2\pi} \|h(r e^{it})\|_X \frac{dt}{2\pi}$$

(see e.g. Blasco and Xu [2, p. 338]). Define now $h(z) = f(Rz)$, and note that $0 \leq r(z) \leq 1$ for all $z \in \mathbb{T}$. Then

$$\begin{aligned}
\int_{\mathbb{T}} \int_{\mathbb{T}} \|f(R_1 z_1 + R_2 z_2)\|_X dz_1 dz_2 &= \int_0^{2\pi} \int_0^{2\pi} \|h(r(e^{is}) e^{it})\|_X \frac{dt}{2\pi} \frac{ds}{2\pi} \\
&\leq \int_0^{2\pi} \int_{\mathbb{T}} \|h(z)\|_X dz \frac{ds}{2\pi} = \int_{\mathbb{T}} \|f(Rz)\|_X dz.
\end{aligned}$$

This completes the proof. \square

A sort of iteration of the preceding result leads to the next

Lemma 2.4. *Let X be a Banach space, and $f : \mathbb{C}^N \rightarrow X$ a holomorphic function. Then for every m*

$$\int_{\mathbb{T}^N} \dots \int_{\mathbb{T}^N} \|f(z^{(1)} + \dots + z^{(m)})\|_X dz^{(1)} \dots dz^{(m)} \leq \int_{\mathbb{T}^N} \|f(mz)\|_X dz.$$

Proof. We fix some m , and do induction with respect to N . For $N = 1$ we obtain from Lemma 2.3 that

$$\begin{aligned} & \underbrace{\int_{\mathbb{T}} \dots \int_{\mathbb{T}}}_{m-2} \int_{\mathbb{T}} \int_{\mathbb{T}} \underbrace{\|f(z^{(1)} + \dots + z^{(m-2)} + z^{(m-1)} + z^{(m)})\|_X}_{=: g_{z^{(1)}, \dots, z^{(m-2)}}(z^{(m-1)} + z^{(m)})} dz^{(m-1)} dz^{(m)} dz^{(1)} \dots dz^{(m-2)} \\ & \leq \underbrace{\int_{\mathbb{T}} \dots \int_{\mathbb{T}}}_{m-2} \int_{\mathbb{T}} \|g_{z^{(1)}, \dots, z^{(m-2)}}(2w)\|_X dw dz^{(1)} \dots dz^{(m-2)} \\ & = \underbrace{\int_{\mathbb{T}} \dots \int_{\mathbb{T}}}_{m-3} \int_{\mathbb{T}} \int_{\mathbb{T}} \|f(z^{(1)} + \dots + z^{(m-2)} + 2w)\|_X dw dz^{(m-2)} dz^{(1)} \dots dz^{(m-3)} \\ & \leq \underbrace{\int_{\mathbb{T}} \dots \int_{\mathbb{T}}}_{m-3} \int_{\mathbb{T}} \|f(z^{(1)} + \dots + z^{(m-3)} + 3w)\|_X dz^{(1)} \dots dz^{(m-3)} dw \\ & \leq \dots \leq \int_{\mathbb{T}} \|f(mz)\|_X dz. \end{aligned}$$

We now assume that the conclusion holds for $N - 1$ and write each $z \in \mathbb{T}^N$ as $z = (u, w)$, with $u \in \mathbb{T}^{N-1}$ and $w \in \mathbb{T}$. Then, using the case $N = 1$ in the first inequality and the inductive hypothesis in the second, we have

$$\begin{aligned} & \int_{\mathbb{T}^N} \dots \int_{\mathbb{T}^N} \|f(z^{(1)} + \dots + z^{(m)})\|_X dz^{(1)} \dots dz^{(m)} \\ & = \int_{\mathbb{T}^{N-1}} \dots \int_{\mathbb{T}^{N-1}} \left(\int_{\mathbb{T}} \dots \int_{\mathbb{T}} \|f((u^{(1)}, w_1) + \dots + (u^{(m)}, w_m))\|_X dw_1 \dots dw_m \right) du^{(1)} \dots du^{(m)} \\ & \leq \int_{\mathbb{T}^{N-1}} \dots \int_{\mathbb{T}^{N-1}} \left(\int_{\mathbb{T}} \|f((u^{(1)}, mw) + \dots + (u^{(m)}, mw))\|_X dw \right) du^{(1)} \dots du^{(m)} \\ & = \int_{\mathbb{T}} \left(\int_{\mathbb{T}^{N-1}} \dots \int_{\mathbb{T}^{N-1}} \|f((u^{(1)}, mw) + \dots + (u^{(m)}, mw))\|_X du^{(1)} \dots du^{(m)} \right) dw \\ & \leq \int_{\mathbb{T}} \left(\int_{\mathbb{T}^{N-1}} \|f((mu, mw) + \dots + (mu, mw))\|_X du \right) dw \\ & = \int_{\mathbb{T}^N} \|f(mz)\|_X dz, \end{aligned}$$

as desired. □

We are now ready to give the *proof of the inequality from Proposition 2.1*. By the polarization formula we know that for every choice of $z_1^{(1)}, \dots, z_m^{(m)} \in \mathbb{T}^N$ we have

$$T(z^{(1)}, \dots, z^{(m)}) = \frac{1}{2^m m!} \sum_{\varepsilon_i = \pm 1} \varepsilon_i \dots \varepsilon_m P\left(\sum_{i=1}^N \varepsilon_i z^{(i)}\right)$$

(see e.g [11] or [12]). Hence we deduce from Lemma 2.4

$$\begin{aligned}
& \int_{\mathbb{T}^N} \dots \int_{\mathbb{T}^N} \|T(z^{(1)}, \dots, z^{(m)})\|_X dz^{(1)} \dots dz^{(m)} \\
& \leq \frac{1}{2^m m!} \sum_{\varepsilon_i = \pm 1} \int_{\mathbb{T}^N} \dots \int_{\mathbb{T}^N} \left\| P\left(\sum_{i=1}^N \varepsilon_i z^{(i)}\right) \right\|_X dz^{(1)} \dots dz^{(m)} \\
& = \frac{1}{2^m m!} \sum_{\varepsilon_i = \pm 1} \int_{\mathbb{T}^N} \dots \int_{\mathbb{T}^N} \left\| P\left(\sum_{i=1}^N z^{(i)}\right) \right\|_X dz^{(1)} \dots dz^{(m)} \\
& = \frac{1}{m!} \int_{\mathbb{T}^N} \dots \int_{\mathbb{T}^N} \left\| P\left(\sum_{i=1}^N z^{(i)}\right) \right\|_X dz^{(1)} \dots dz^{(m)} \\
& \leq \frac{1}{m!} \int_{\mathbb{T}^N} \|P(mz)\|_X dz = \frac{m^m}{m!} \int_{\mathbb{T}^N} \|P(z)\|_X dz.
\end{aligned}$$

Then by Lemma 2.2 we obtain

$$\begin{aligned}
\left(\sum_{i_1, \dots, i_m}^N \|a_{i_1, \dots, i_m}\|_X^q \right)^{1/q} & \leq (C_q(X)K)^m \int_{\mathbb{T}^\infty} \dots \int_{\mathbb{T}^\infty} \|T(z^{(1)}, \dots, z^{(m)})\|_X dz^{(1)} \dots dz^{(m)} \\
& = (C_q(X)K)^m \frac{m^m}{m!} \int_{\mathbb{T}^N} \|P(z)\|_X dz,
\end{aligned}$$

which completes the proof of Proposition 2.1. \square

A second proposition is needed which allows to reduce the proof of our main result 1.1 to the homogeneous case. It is a vector-valued version of a result of [6, Theorem 9.2] with a similar proof (here only given for the sake of completeness).

Proposition 2.5. *There is a contractive projection*

$$\Phi_m : H_p(\mathbb{T}^N, X) \rightarrow H_p(\mathbb{T}^N, X), \quad f \mapsto f_m,$$

such for all $f \in H_p(\mathbb{T}^N, X)$

$$\hat{f}(\alpha) = \hat{f}_m(\alpha) \text{ for all } \alpha \in \mathbb{N}_0^N \text{ with } |\alpha| = m. \quad (9)$$

Proof. Let $\mathcal{P}(\mathbb{C}^N, X) \subset H_p(\mathbb{T}^N, X)$ be the subspace all finite polynomials $f = \sum_{\alpha \in \Lambda} c_\alpha z^\alpha$; here Λ is a finite set of multi indices in \mathbb{N}_0^N and the coefficients $c_\alpha \in X$. Define the linear projection Φ_m^0 on $\mathcal{P}(\mathbb{C}^N, X)$ by

$$\Phi_m^0(f)(z) = f_m(z) = \sum_{\alpha \in \Lambda, |\alpha|=m} \hat{f}(\alpha) z^\alpha;$$

clearly, we have (9). In order to show that Φ_m^0 is a contraction on $(\mathcal{P}(\mathbb{C}^N, X), \|\cdot\|_p)$ fix some function $f \in \mathcal{P}(\mathbb{C}^N, X)$ and $z \in \mathbb{T}^N$, and define

$$f(z \cdot) : \mathbb{T} \rightarrow X, \quad w \mapsto f(zw).$$

Clearly, we have

$$f(zw) = \sum_k f_k(z) w^k,$$

and hence

$$f_m(z) = \int_{\mathbb{T}} f(zw) w^{-m} dw.$$

Integration, the continuous Minkowski inequality and the rotation invariance of the normalized Lebesgue measure on \mathbb{T}^N give

$$\begin{aligned} \int_{\mathbb{T}^N} \|f_m(z)\|_X^p dz &= \int_{\mathbb{T}^N} \left\| \int_{\mathbb{T}} f(zw) w^{-m} dw \right\|_X^p dz \\ &\leq \int_{\mathbb{T}^N} \left(\int_{\mathbb{T}} \|f(zw)\|_X dw \right)^p dz \leq \int_{\mathbb{T}} \int_{\mathbb{T}^N} \|f(zw)\|_X^p dz dw = \int_{\mathbb{T}^N} \|f(z)\|_X^p dz, \end{aligned}$$

which proves that Φ_m^0 is a contraction on $(\mathcal{D}(\mathbb{C}^N, X), \|\cdot\|_p)$. By Fejer's theorem (vector-valued) we know that $\mathcal{D}(\mathbb{C}^N, X)$ is a dense subspace of $H_p(\mathbb{T}^N, X)$. Hence Φ_m^0 extends to a contractive projection Φ_m on $H_p(\mathbb{T}^N, X)$. This extension Φ_m still satisfies (9) since for each multi index α the mapping $H_p(\mathbb{T}^N, X) \rightarrow X$, $f \mapsto \hat{f}(\alpha)$ is continuous. \square

3 Proof of the main result

We are now ready to prove Theorem 1.1. Let $1 \leq p < \infty$, and recall from (1) that

$$1 - \frac{1}{\text{Cot}(X)} = S_\infty(X) \leq S_p(X);$$

see Remark 3.1 for a direct argument. Hence it suffices to concentrate on the upper estimate in Theorem 1.1: Since we obviously have $S_p(X) \leq S_1(X)$, we are going to prove that

$$S_1(X) \leq 1 - \frac{1}{\text{Cot}(X)}. \quad (10)$$

Suppose first that X has no finite cotype. For $D = \sum_n a_n n^{-s} \in \mathcal{H}_1(X)$ we take $f \in H_1(\mathbb{T}^\infty, X)$ with $D = \mathfrak{B}_X f$. Note that

$$|\hat{f}(\alpha)| \leq \int_{\mathbb{T}^\infty} |f(w) w^{-\alpha}| dw = \|f\|_{L_1(\mathbb{T}^\infty, X)} < \infty$$

and, by the definition of \mathfrak{B}_X , the coefficients of D are also bounded by $\|f\|_{L_1(\mathbb{T}^\infty, X)}$. As a consequence,

$$\sum_{n=1}^{\infty} \|a_n\|_X \frac{1}{n^s} \leq \sum_{n=1}^{\infty} \|f\|_{L_1(\mathbb{T}^\infty, X)} \frac{1}{n^s} < \infty$$

whenever $\text{Re } s > 1$. This means that $S_1(X) \leq 1$ and gives (10) for $\text{Cot}(X) = \infty$.

Now if X has finite cotype, take $q > \text{Cot}(X)$ and $\varepsilon > 0$, and put $s = (1 - \frac{1}{q})(1 + 2\varepsilon)$. Choose an integer k_0 such $p_{k_0}^{\varepsilon/q'} > eC_q(X)K \sum_{j=1}^{\infty} \frac{1}{p_j^{1+\varepsilon}}$, and define

$$\tilde{p} = (\underbrace{p_{k_0}, \dots, p_{k_0}}_{k_0 \text{ times}}, p_{k_0+1}, p_{k_0+2}, \dots).$$

We are going to show that there is a constant $C(q, X, \varepsilon) > 0$ such that for every $f \in H_1(\mathbb{T}^\infty, X)$ we have

$$\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} \|\hat{f}(\alpha)\|_X \frac{1}{\tilde{p}^{s\alpha}} \leq C(q, X, \varepsilon) \|f\|_{H_1(\mathbb{T}^\infty, X)}. \quad (11)$$

This finishes the argument: By Remark 1.2 the sequence $1/p^s \in \text{mon } H_1(\mathbb{T}^\infty, X)$. But in view of Bohr's transform from (6), this means that for every Dirichlet series $D = \sum_n a_n n^{-s} = \mathfrak{B}_X f \in \mathcal{H}_1(X)$ with $f \in H_1(\mathbb{T}^\infty, X)$ we have

$$\sum_{n=1}^{\infty} \|a_n\|_X \frac{1}{n^s} = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} \|\hat{f}(\alpha)\|_X \frac{1}{p^{s\alpha}} < \infty.$$

Therefore $\sigma_a(D) \leq (1 - \frac{1}{q})(1 + 2\varepsilon)$ for each such D which, since $\varepsilon > 0$ was arbitrary, is what we wanted to prove.

It remains to check (11); the idea is to show first that (11) holds for all X -valued H_1 -functions which only depend on N variables: There is a constant $C(q, X, \varepsilon) > 0$ such that for all N and every $f \in H_1(\mathbb{T}^N, X)$ we have

$$\sum_{\alpha \in \mathbb{N}_0^N} \|\hat{f}(\alpha)\|_X \frac{1}{\tilde{p}^{s\alpha}} \leq C(q, X, \varepsilon) \|f\|_{H_1(\mathbb{T}^N, X)}. \quad (12)$$

In order to understand that (12) implies (11) (and hence the conclusion), assume that (12) holds and take some $f \in H_1(\mathbb{T}^\infty, X)$. Given an arbitrary N , define

$$f_N : \mathbb{T}^N \rightarrow X, \quad f_N(w) = \int_{\mathbb{T}^\infty} f(w, \tilde{w}) d\tilde{w}.$$

Then it can be easily shown that $f_N \in L_1(\mathbb{T}^N, X)$, $\|f_N\|_1 \leq \|f\|_1$, and $\hat{f}_N(\alpha) = \hat{f}(\alpha)$ for all $\alpha \in \mathbb{Z}^N$. If we now apply (12) to this f_N , we get

$$\sum_{\alpha \in \mathbb{N}_0^N} \|\hat{f}(\alpha)\|_X \frac{1}{\tilde{p}^{s\alpha}} \leq C(q, X, \varepsilon) \|f\|_{H_1(\mathbb{T}^\infty, X)},$$

which, after taking the supremum over all possible N on the left side, leads to (11).

We turn to the proof of (12), and here in a first step will show the following: For every N , every m -homogeneous polynomial $P : \mathbb{C}^N \rightarrow X$ and every $u \in \ell_{q'}$ we have

$$\sum_{\substack{\alpha \in \mathbb{N}_0^N \\ |\alpha|=m}} \|\hat{P}(\alpha) u^\alpha\|_X \leq (eC_q(X)K)^m \int_{\mathbb{T}^N} \|P(z)\|_X dz \left(\sum_{j=1}^{\infty} |u_j|^{q'} \right)^{m/q'}. \quad (13)$$

Indeed, take such a polynomial $P(z) = \sum_{\alpha \in \mathbb{N}_0^N, |\alpha|=m} \hat{P}(\alpha) z^\alpha$, $z \in \mathbb{T}^N$, and look at its unique m -linear symmetrization

$$T : \mathbb{C}^N \times \dots \times \mathbb{C}^N \rightarrow X, \quad T(z^{(1)}, \dots, z^{(m)}) = \sum_{i_1, \dots, i_m=1}^N a_{i_1, \dots, i_m} z_{i_1}^{(1)}, \dots, z_{i_m}^{(m)}.$$

Then we know from Proposition 2.1 that

$$\left(\sum_{i_1, \dots, i_m=1}^N \|a_{i_1, \dots, i_m}\|_X^q \right)^{1/q} \leq (eC_q(X)K)^m \int_{\mathbb{T}^N} \|P(z)\|_X dz.$$

Hence (13) follows by Hölder's inequality:

$$\begin{aligned} \sum_{\alpha \in \mathbb{N}_0^N, |\alpha|=m} \|\hat{P}(\alpha)u^\alpha\|_X &= \sum_{i_1, \dots, i_m=1}^N \|a_{i_1, \dots, i_m}\|_X |u_{i_1} \dots u_{i_m}| \\ &\leq (eC_q(X)K)^m \int_{\mathbb{T}^N} \|P(z)\|_X dz \left(\sum_{j=1}^{\infty} |u_j|^{q'} \right)^{m/q'}. \end{aligned}$$

We finally give the proof of (12): Take $f \in H_1(\mathbb{T}^N, X)$, and recall from Proposition 2.5 that for each integer m there is an m -homogeneous polynomial $P_m : \mathbb{C}^N \rightarrow X$ such that $\|P_m\|_{H_1(\mathbb{T}^N, X)} \leq \|f\|_{H_1(\mathbb{T}^N, X)}$ and $\hat{P}_m(\alpha) = \hat{f}(\alpha)$ for all $\alpha \in \mathbb{N}_0^N$ with $|\alpha| = m$. Finally, from (13), the definition of s , and the fact that $\max\{p_{k_0}, p_j\} \leq \tilde{p}_j$ for all j we conclude that

$$\begin{aligned} \sum_{\alpha \in \mathbb{N}_0^N} \|\hat{f}(\alpha)\|_X \frac{1}{\tilde{p}^{s\alpha}} &= \sum_{m=1}^{\infty} \sum_{\alpha \in \mathbb{N}_0^N, |\alpha|=m} \|\hat{P}_m(\alpha)\|_X \frac{1}{\tilde{p}^{s\alpha}} \\ &\leq \sum_{m=1}^{\infty} (eC_q(X)K)^m \|P_m\|_{H_1(\mathbb{T}^N, X)} \left(\sum_{j=1}^{\infty} \frac{1}{\tilde{p}_j^{sq'}} \right)^{m/q'} \\ &= \sum_{m=1}^{\infty} (eC_q(X)K)^m \|f\|_{H_1(\mathbb{T}^N, X)} \left(\sum_{j=1}^{\infty} \frac{1}{\tilde{p}_j^{1+2\varepsilon}} \right)^{m/q'} \\ &= \sum_{m=1}^{\infty} (eC_q(X)K)^m \|f\|_{H_1(\mathbb{T}^N, X)} \left(\sum_{j=1}^{\infty} \frac{1}{\tilde{p}_j^{1+\varepsilon}} \frac{1}{\tilde{p}_j^\varepsilon} \right)^{m/q'} \\ &\leq \|f\|_{H_1(\mathbb{T}^N, X)} \sum_{m=1}^{\infty} \underbrace{\left(\frac{eC_q(X)K \left(\sum_{j=1}^{\infty} \frac{1}{\tilde{p}_j^{1+\varepsilon}} \right)^{1/1+\varepsilon}}{p_{k_0}^{\varepsilon/q'}} \right)^m}_{\leq 1}. \end{aligned}$$

This completes the proof of Theorem 1.1. \square

Remark 3.1. We end this note with a direct proof of the fact

$$1 - \frac{1}{\text{Cot}(X)} \leq S_p(X), \quad 1 \leq p < \infty \quad (14)$$

in which we do not use the inequality

$$1 - \frac{1}{\text{Cot}(X)} \leq S_\infty(X) \quad (15)$$

from [8] (here repeated in (1)). The proof of (15) given in [8] in a first step shows that $1 - 1/\Pi(X) \leq S_\infty(X)$ where

$$\Pi(X) = \inf\{r \geq 2 \mid \text{id}_X \text{ is } (r, 1)\text{-summing}\},$$

and then, in a second step, applies a fundamental theorem of Maurey and Pisier stating that $\Pi(X) = \text{Cot}(X)$.

The following argument for (14) is very similar to the original one from [8] but does not use the Maurey-Pisier theorem (since we here consider $\mathcal{H}_p(X)$, $1 \leq p < \infty$ instead of $\mathcal{H}_\infty(X)$): By the proof of Corollary 1.3, inequality (14) is equivalent to

$$M_p(X) \leq \frac{\text{Cot}(X)}{\text{Cot}(X) - 1}.$$

Take $r < M_p(X)$, so that $\ell_r \cap B_{c_0} \subset \text{mon } H_p(\mathbb{T}^\infty, X)$. Let $H_p^1(\mathbb{T}^\infty, X)$ be the subspace of $H_p(\mathbb{T}^\infty, X)$ formed by all 1-homogeneous polynomials (i.e., linear operators). We can define a bilinear operator $\ell_r \times H_p^1(\mathbb{T}^\infty, X) \rightarrow \ell_1(X)$ by $(z, f) \mapsto (z_j f(e_j))_j$ which, by a closed graph argument, is continuous. Therefore, there is a constant M such that for all $z \in \ell_r$ and all $f \in H_p^1(\mathbb{T}^\infty, X)$ we have

$$\sum_j |z_j| \|f(e_j)\|_X \leq M \|z\|_{\ell_r} \|f\|_{H_p(\mathbb{T}^\infty, X)}.$$

Taking the supremum over all $z \in B_{\ell_r}$ we obtain for all $f \in H_p^1(\mathbb{T}^\infty, X)$

$$\left(\sum_j \|f(e_j)\|_X^{r'} \right)^{1/r'} \leq M \|f\|_{H_p(\mathbb{T}^\infty, X)}.$$

Now, take $x_1, \dots, x_N \in X$ and define $f \in H_p^1(\mathbb{T}^\infty, X)$ by $f(e_j) = x_j$ if $1 \leq j \leq N$, $f(e_j) = 0$ if $j > N$ and extend it by linearity. By the previous inequality and Lemma 2.5 we have

$$\left(\sum_{j=1}^N \|x_j\|_X^{r'} \right)^{1/r'} \leq M \left(\int_{\mathbb{T}^N} \left\| \sum_{j=1}^N x_j z_j \right\|_X^{r'} dz \right)^{1/r'}.$$

By Kahane's inequality, X has cotype r' , which means that $r' > \text{Cot}(X)$ or, equivalently, $r < \frac{\text{Cot}(X)}{\text{Cot}(X)-1}$. Since $r < M_p(X)$ was arbitrary, we obtain (14).

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